LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – STATISTICS

FIRST SEMESTER – NOVEMBER 2007

ST 1808 - ANALYSIS

BB 12

Date : 25/10/2007 Dept. No. Max. : 100 Marks
Time : 1:00 - 4:00

SECTION – A

Answer ALL questions . (10 x 2 = 20 marks)

1. Define a metric on R^n and verify that it satisfies the conditions for a metric.
2. Let (X , ρ) be a metric space . Then show that a sequence in X cannot converge to two limits.
3. Let ρ and σ be two metrics on X. Then show that ρ and σ are equivalent if ∃ positive constants λ and μ such that $\lambda \rho \le \sigma \le \mu \rho$.
4. Let (X , ρ) be a metric space . Obtain the open ball when (i) $X = R^1$ (ii) $X = R^2$ and (iii) $X = R^3$.
5. State any three properties of a linear function.
6. Define a contraction mapping and show that it is continuous.
7. Give an example of sequences $\{ x_n \}$ and $\{ v_n \}$ satisfying the relations: (i) $x_n = O(v_n)$ (ii) $x_n = o(v_n)$ and (iii) $x_n \sim v_n$.
8. State D`Alemberts ratio test for the convergence of a series of complex terms.
9. Illustrate that pointwise convergence does not imply uniform convergence of a sequence of functions.
10. Show that if $f_1(x) \le f_2 (x)$; $a \le x \le b$, then

$$\int_a^b f_1 \, dg \le \int_a^b f_2 \, dg .$$

SECTION – B

Answer any FIVE questions . (5 x 8 = 40 marks)

11. Let $X = R^2$. Take $x_n = (3n / (2n + 1) ,\ 2n^2 / (n^2 – 2))$; n = 1,2,…. Show that (i) $x_n \to (½ , 2)$ as $n \to \infty$.
(ii) $x_n \to (3/2 , 2)$ as $n \to \infty$.
12. Prove the following:
(i) If G in X is open, then G′ is closed.
(ii) If F in X is closed, Then F′ is open.
13. Let (X , ρ) be any metric space and $a \in X$ be fixed.
Define $g : X \to R^1$ as $g(x) = \rho (a , x)$; $x \in X$.
Then prove that g is continuous on X.
14. Let (X , ρ) be any metric space and let f_i , i = 1,2,…, n be functions from X to R^1 . Define $f = (f_1 , \dots , f_n) : X \to R^n$
as $f(x) = (f_1 (x), \dots , f_n (x))$. Then show that f is continuous at $x_0 \in X$ iff f_i is continuous at x_0 ∀ i = 1,2, … , n.
15. State and prove Banach’s fixed point theorem regarding contraction mapping.

16. Prove that the number Λ is the upper limit of $\{ x_n \}$ iff given $\varepsilon > 0$
   (i) $x_n < \Lambda + \varepsilon$ for sufficiently large n.
   (ii) $x_n > \Lambda - \varepsilon$ for infinitely many n.

17. Give examples to show that if $\limsup_{n \to \infty} | u_n |^{1/n} = 1$, then the series may converge or diverge.

18. If $f : X \to R^n$ ($X \subset R^m$) is differentiable at $\xi \in X$, then show that f is continuous at ξ.

SECTION – C

Answer any TWO questions . (2 x 20 = 40 marks)

19.(a) Prove the following:
   (i) The intersection of a finite collection of open sets is open.
   (ii) The union of a finite collection of closed sets is closed. (10)

(b) Let $X = R^2$, $E = R^2 - \{ (0,0) \}$, $Y = R^1$. Define $g : E \to R^1$ as $g(x, y) = x^2 / (x^2 + y^2)$; $(x, y) \in E$.
Show that $\lim_{(x,y) \to (0,0)} g(x, y)$ does not exist. (10)

20. (a) Let V , W be the normed vector spaces. Let $f : V \to W$ be a linear transformation. Then prove that the following three statements are equivalent :
   (i) f is continuous on V.
   (ii) There exists a point x_o in V at which f is continuous.
   (iii) $\| f(x) \| / \| x \|$ is bounded for $x \in V - \{ \theta \}$. (16)

(b) State any two properties of compact sets. (4)

21. (a) If the series $\sum u_n$ of real / complex valued functions converges uniformly to s on (X , ρ) and if each u_n is continuous at c , then prove that s is continuous at c . (10)

(b) Prove the series
$$\sum_{n=1}^{\infty} (-1)^{n+1} / (n + x)$$ converges uniformly on $[0, \infty)$ but that it does not converge absolutely for any x. (10)

22. (a) Let $f \in R(g ; a, b)$ and define F on [a , b] as
$$F(x) = \int_a^x f \, dg ; \quad x \in [a, b].$$
Prove that
   (i) F is continuous at every point of continuity of g.
   (ii) F is differentiable at a point where f is continuous and g is differentiable. At such a point c , show that $F'(c) = f(c) \, g'(c)$. (16)

(b) Let $f : R^m \to R^n$ be linear. Show that the function is differentiable and the linear derivative $D f(\xi) = f$. (4)
